Propagation of Particles Due to an Explosion in Elliptic Orbit

Joshua Ashenberg*
Chelmsford, Massachusetts 01863

Introduction

IN recent years, the short-term propagation of orbital debris has been modeled[1,2] mainly by applying the relative motion along a circular orbit.[3] Although some methods consider eccentric orbits by using series expansions,[4] these lack the compactness of a closed-form analytical solution. This Note is motivated by the leading publications on the elliptic relative motion: Brumberg,[5] Danby,[6] and Broucke.[7] Because of the uncertainty in the initial conditions, a model for error propagation is essential. Brumberg[5] presented the general idea but did not give an explicit solution. A recent paper[8] suggested a solution to the propagation of the covariance along circular orbits. In the present Note, the simple form of the isotropic initial velocities distribution is used to form a closed-form solution for the volume as well as the covariance propagation along a general elliptic orbit.

Formulation

In the present section the state transition matrix for the relative motion is constructed. The main assumption is that of linearity, i.e., a small spread from the parent orbit.

Let r_N and V_N be the radius vector and velocity of the nominal particle, respectively, and r_j and V_j be the radius vector and velocity of a particle from the explosion, respectively. The linear equations of motion can be derived from the quadratic part of the Lagrangian,[9] expressed in terms of the cylindrical coordinates $\{r, \theta, z\}$. The resulting equations of relative motion for the particle j are linear and time dependent (r_N and $\dot{\theta}_N$ are functions of time). In the case of a circular orbit, these equations reduce to the Clohessy-Wiltshire equations.[3]

The solution can be written in terms of the 6×6 transition matrix $\Phi(t, t_0)$, where t_0 and t are the breakup time and the current time, respectively. Thanks to the uncoupling between the orbital plane (x, y) and the orthogonal direction z, the transition matrix partitions for the orbital plane and the orthogonal solution are obtained separately. Let us first evaluate the transition matrix in the orbital plane (4×4). The state variables and the orbital elements (semimajor axis, eccentricity, mean anomaly at epoch, and argument of periapsis) are $X = \{r, \theta, \dot{r}, \dot{\theta}\}$ and $a_\alpha = \{a, e, M_0, \omega\}$, respectively. The objective is to map the variation in orbital elements into the relative coordinates. The relation between a particular relative state $\{x, y, \dot{x}, \dot{y}\}_j$ and the corresponding variation is given by the following kinematical relation:

$$\begin{pmatrix} x \\ y \\ \dot{x} \\ \dot{y} \end{pmatrix}_j = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & r & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \dot{r} & 0 & r \end{pmatrix} \begin{pmatrix} \dfrac{\partial r}{\partial a_\alpha} \\ \dfrac{\partial \theta}{\partial a_\alpha} \\ \dfrac{\partial \dot{r}}{\partial a_\alpha} \\ \dfrac{\partial \dot{\theta}}{\partial a_\alpha} \end{pmatrix} \delta a_{\alpha_j} \equiv M\left(\frac{\partial X}{\partial a_\alpha}\right)\delta a_{\alpha_j} \equiv \Psi \delta a_{\alpha_j} \tag{1}$$

where Ψ is the fundamental matrix of the relative motion. Now the evaluation of the transition matrix is a straightforward procedure yielding

$$\Phi(t, t_0) = \Psi(t)\Psi(t_0)^{-1} = M\left(\frac{\partial X}{\partial a_\alpha}\right)\left(\frac{\partial a_\alpha}{\partial X}\right)_0 M_0^{-1} \tag{2}$$

where the subscript 0 indicates the initial time. The orthogonal component is governed by the following second-order differential equation:

$$\ddot{\phi}_{36} = -\frac{\mu}{r^3}\phi_{36}, \qquad \phi_{36}(t_0) = 0, \qquad \dot{\phi}_{36}(t_0) = 1 \tag{3}$$

where μ is the geocentric gravitational constant.

The solution is the well-known g function.[10] It can be expressed as a time series or as the following closed-form expression in the true anomaly θ:

$$g = \frac{r r_0}{\bar{\omega} a^2 \sqrt{1 - e^2}} \sin(\theta - \theta_0) \tag{4}$$

where $\bar{\omega}$ is the orbital rate.

Only one partition of the transition matrix will be needed for further applications:

$$\Phi_{12}(t, t_0) = \begin{bmatrix} \displaystyle\sum_\alpha \frac{\partial r}{\partial a_{\alpha(t)}} \frac{\partial a_\alpha}{\partial \dot{r}_{(t_0)}} & \displaystyle\frac{1}{r_0}\sum_\alpha \frac{\partial r}{\partial a_{\alpha(t)}} \frac{\partial a_\alpha}{\partial \dot{\theta}_{(t_0)}} & 0 \\ \displaystyle r\sum_\alpha \frac{\partial \theta}{\partial a_{\alpha(t)}} \frac{\partial a_\alpha}{\partial \dot{r}_{(t_0)}} & \displaystyle\frac{r}{r_0}\sum_\alpha \frac{\partial \theta}{\partial a_{\alpha(t)}} \frac{\partial a_\alpha}{\partial \dot{\theta}_{(t_0)}} & 0 \\ 0 & 0 & g(t, t_0) \end{bmatrix} \tag{5}$$

The partials in the transition matrix are computed directly from the formulation of the two-body problem in orbital elements.

Volume of the Particles Cloud

The volume propagation is based on the mapping $\{r\} = \Phi_{12}(t, t_0)\{\dot{r}_0\}$. It can be shown that the volume is the image of its initial value under the mapping of the system transition matrix.[11] In this case, $\Phi_{12}(t, t_0) = \partial r / \partial \dot{r}_0$ maps the velocity impulse into the volume, by the factor $|\Phi_{12}|$. The results will be presented in terms of the nondimensional volume[1] $\tilde{\text{Vol}}$, and the nondimensional time $\tau = \bar{\omega} t / 2\pi$:

$$\tilde{\text{Vol}} = \frac{4\pi}{3}\frac{1 + e\cos\nu_0}{1 + e\cos\nu}\left[\sum_\alpha \frac{\partial r}{\partial a_{\alpha(t)}} \frac{\partial a_\alpha}{\partial \dot{r}_{(t_0)}} \sum_\alpha \frac{\partial \theta}{\partial a_{\alpha(t)}} \frac{\partial a_\alpha}{\partial \dot{\theta}_{(t_0)}} - \sum_\alpha \frac{\partial r}{\partial a_{\alpha(t)}} \frac{\partial a_\alpha}{\partial \dot{\theta}_{(t_0)}} \sum_\alpha \frac{\partial \theta}{\partial a_{\alpha(t)}} \frac{\partial a_\alpha}{\partial \dot{r}_{(t_0)}}\right] \cdot g(\nu, \nu_0) \tag{6}$$

The volume consists of two independent variables: the explicit time and the true anomaly. It can be easily formulated as a function of the true anomaly alone. However, to express it as a unique function of the time, Kepler's equation can be solved, or the true anomaly can be presented as a power series in eccentricity.[12]

The influence of the breakup location is demonstrated in Fig. 1. The outcome of explosions at the perigee and the apogee deviates as the eccentricity increases. Moreover, an explosion at the apogee gives rise to higher volume amplitude. Figure 2 presents the volume as a function of the true anomaly for $e = 0.5$. The main contributor to the magnitude of the volume is the amplitude of g. It is three times bigger for an apogee breakup than for a similar condition in perigee. An interesting phenomenon in the area propagation is that the highest expansion occurs after the apogee passage, whereas the highest compression is before its passage. This is somewhat analogous to the motion of a fluid before and after a stagnation point.

Covariance Propagation of the Particles Cloud

The covariance propagation due to the uncertainty in the isotropic explosion will be examined. Because of the linearity of the rel-

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*Aerospace Consultant, P.O. Box 606. Member AIAA.

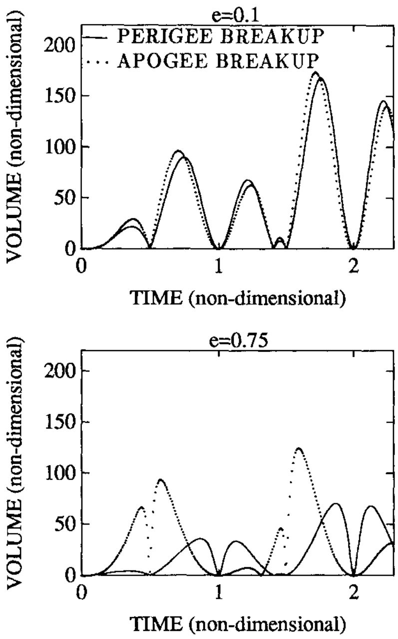

Fig. 1 Volume due to breakup at perigee/apogee.

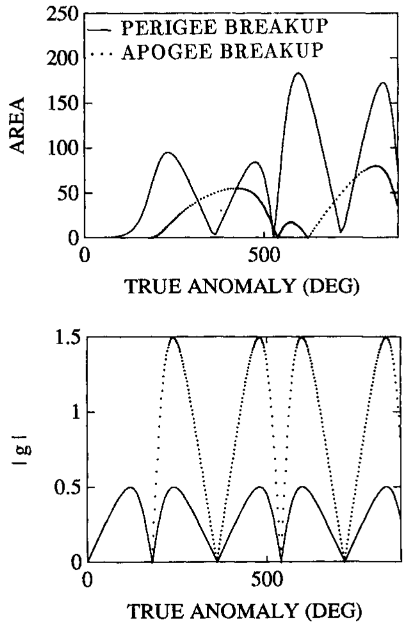

Fig. 2 Area and the *g* function.

ative motion, the uncertainty propagation can be modeled by applying the Riccati equation for the covariance. The average is zero, and the initial probability density function is arbitrary.[13] An analytical solution of the Riccati equation for our case is straightforward. The propagation of covariance due to an isotropic explosion can be computed as $P_{xx}(t) = \Phi_{12}(t, t_0) P_{\dot{x}_0 \dot{x}_0} \Phi_{12}^T(t, t_0)$, with the following initial conditions: $P_{\dot{x}_0 \dot{x}_0} = \mathrm{diag}\ (\sigma_V^2)$, where σ_V^2 is the variance of the explosion impulse. The explicit expression for the covariance components turns out to be

$$P_{xx} = \left\{ \left[\sum_{\alpha} \frac{\partial r}{\partial a_{\alpha(t)}} \frac{\partial a_{\alpha}}{\partial \dot{r}_{(t_0)}} \right]^2 + \left[\frac{1}{r_0} \sum_{\alpha} \frac{\partial r}{\partial a_{\alpha(t)}} \frac{\partial a_{\alpha}}{\partial \theta_{(t_0)}} \right]^2 \right\} \cdot \sigma_V^2$$

$$P_{yy} = \left\{ \left[r \sum_{\alpha} \frac{\partial \theta}{\partial a_{\alpha(t)}} \frac{\partial a_{\alpha}}{\partial \dot{r}_{(t_0)}} \right]^2 + \left[\frac{r}{r_0} \sum_{\alpha} \frac{\partial \theta}{\partial a_{\alpha(t)}} \frac{\partial a_{\alpha}}{\partial \theta_{(t_0)}} \right]^2 \right\} \cdot \sigma_V^2 \tag{7}$$

$$P_{zz} = g(t, t_0)^2 \cdot \sigma_V^2$$

The probability ellipsoid can be constructed from the eigensystem of the covariance. It gives an idea about the size of the spread as well as the orientation of the debris cloud with respect to the orbit.

The restriction of the breakup to an isotropic explosion type leads to a simple relation between the volume and the covariance. Expressing the determinant of the transition matrix as $|P_{xx}| = |\det(\Phi_{12})|^2 \sigma_V^6$ and applying the property $|P_{xx}| = \sigma_1^2 \times \sigma_2^2 \times \sigma_3^2$, where σ_1^2, σ_2^2, and σ_3^2 are the eigenvalues of P_{xx} and $\sigma_3 = \sigma_V \times g$, one finds that the volume is

$$\tilde{\mathrm{V}}\mathrm{ol} = \frac{4\pi}{3} \left| \frac{\sigma_1 \times \sigma_2 \times g}{\sigma_V^2} \right| \tag{8}$$

This relation clearly shows the similarity between the volume and the covariance. Note that the product $\sigma_1 \times \sigma_2$ reflects the area in the orbital plane.

In the particular case of a circular orbit, the following explicit time-dependence relations are obtained:

$$P_{xx}(t) \equiv \sigma_x^2 = [\sin^2 t + (2 - 2\cos t)^2] \cdot \sigma_V^2$$

$$P_{yy}(t) \equiv \sigma_y^2 = [(2\cos t - 2)^2 + (4\sin t - 3t)^2] \cdot \sigma_V^2 \tag{9}$$

$$P_{zz}(t) \equiv \sigma_z^2 = \sin^2 t \cdot \sigma_V^2$$

The radial variance is periodic, whereas the transverse variance has a secular growth. It follows that the inclination of the probability ellipsoid relative to the orbit decreases with time.

Summary

Closed-form analytical expressions for the propagation of the volume and the covariance of orbital debris have been presented. The volume development is found to be very sensitive to the breakup location. The amplification peaks are maximal when the breakup occurs at the apogee. The orthogonal motion reflects a major difference between the two extreme cases; the apogee explosion gives rise to an amplification factor of $(1 + e)/(1 - e)$ relative to the explosion at perigee. A similarity between the volume and the variances is shown.

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